MULTIPLICATIVE DIOPHANTINE EXPONENTS OF HYPERPLANES AND THEIR NONDEGENERATE SUBMANIFOLDS

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Abstract. We consider multiparameter dynamics on the space of unimolular lattices. Along with quantitative nondivergence we prove that multiplicative Diophantine exponents of hyperplanes are inherited by their nondegenerate submanifolds.

1. Introduction

Given any $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we define its Diophantine exponent as

(1.1)
$$\omega(\mathbf{y}) = \sup\left\{v \mid \exists\, \infty \text{ many } \mathbf{q} \in \mathbb{Z}^n \text{ with } |\langle \mathbf{q}, \mathbf{y}\rangle + p| < \|\mathbf{q}\|^{-v} \text{ for some } p \in \mathbb{Z}\right\},$$

where $\langle \mathbf{q}, \mathbf{y}\rangle$ stands for the inner product of vectors $\mathbf{q}$ and $\mathbf{y}$.

Remark 1.1. *In* (1.1), $\|\cdot\|$ *can be any norm on* $\mathbb{R}^n$. *Same in* (1.12).

It can be deduced from Dirichlet's Theorem [3] that

$$\omega(\mathbf{y}) \geq n \quad \forall \mathbf{y} \in \mathbb{R}^n. \tag{1.2}$$

We call $\mathbf{y}$ very well approximable (abbreviated as VWA) if $\omega(\mathbf{y}) > n$. It is known that the set of VWA vectors has zero Lebesgue measure. Following [7] the Diophantine exponent $\omega(\mu)$ of a Borel measure μ is set to be the μ-essential supremum of the ω function, that is,

$$\omega(\mu) = \sup\left\{v \mid \mu\{\mathbf{y} \mid \omega(\mathbf{y}) > v\} > 0\right\}. \tag{1.3}$$

Let $\mathcal{M}$ be a smooth submanifold of $\mathbb{R}^n$ and μ be the measure class of the Riemannian volume on $\mathcal{M}$. More precisely put, let μ be the pushforward $\mathbf{f}_*\lambda$ of λ(the Lebesgue measure) by any smooth map $\mathbf{f}$ parameterizing $\mathcal{M}$. Then the Diophantine exponent of $\mathcal{M}$, which we denote by $\omega(\mathcal{M})$, is set to be equal to $\omega(\mu)$. $\mathcal{M}$ is called extremal if $\omega(\mathcal{M}) = n$, that is, almost all points of $\mathcal{M}$ are not VWA. A trivial example of an extremal submanifold of $\mathbb{R}^n$ is $\mathbb{R}^n$ itself.

K. Mahler [5] conjectured in 1932 that

$$\mathcal{M} = \left\{ (x, x^2, \ldots, x^n) \middle| \, x \in \mathbb{R}\right\} \tag{1.4}$$

is an extremal submanifold. This was proved by Sprindžuk [12] in 1964. The curve (1.4) has a notable property that it does not lie in any proper affine subspace of $\mathbb{R}^n$. We might describe and formalize this property in terms of nondegeneracy condition as follows. Let $\mathbf{f} = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ be a differentiable map where U is an open subset of $\mathbb{R}^d$. $\mathbf{f}$ is called nondegenerate in an affine subspace $\mathcal{L}$ of $\mathbb{R}^n$ at $\mathbf{x} \in U$ if $\mathbf{f}(U) \subset \mathcal{L}$ and the span of all the partial derivatives of $\mathbf{f}$ at $\mathbf{x}$ up to some order

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coincides with the linear part of $\mathcal{L}$. If $\mathcal{M}$ is a d dimensional submanifold of $\mathcal{L}$ we will say that $\mathcal{M}$ is nondegenerate in $\mathcal{L}$ at $\mathbf{y} \in \mathcal{M}$ if some diffeomorphism of $\mathbf{f}$ between an open subset U of $\mathbb{R}^d$ and a neighborhood of $\mathbf{y}$ in $\mathcal{M}$ is nondegenerate in $\mathcal{L}$ at $\mathbf{f}^{-1}(\mathbf{y})$. We will say $\mathcal{M}$ is nondegenerate in $\mathcal{L}$ if it is nondegenerate in $\mathcal{L}$ at almost all points of $\mathcal{M}$.

It was conjectured by Sprindžuk [13] in 1980 that almost all points on a nondegenerate analytic submanifold of $\mathbb{R}^n$ are not very well approximable. In 1998 D. Kleinbock and G.A. Margulis proved in [9]

Theorem 1.2. *Let $\mathcal{M}$ be a smooth nondegenerate submanifold of $\mathbb{R}^n$, then $\mathcal{M}$ is extremal.*

[6] studied the conditions under which an affine subspace $\mathcal{L}$ of $\mathbb{R}^n$ is extremal and showed that $\mathcal{L}$ is extremal if and only if its nondegenerate submanifolds are extremal. [7] derived formulas for computing $\omega(\mathcal{L})$ and $\omega(\mathcal{M})$ when $\mathcal{L}$ is not extremal and $\mathcal{M}$ is an arbitrary nondegenerate submanifold in it. This breakthrough was achieved through sharpening of some nondivergence estimates in the space of unimodular lattices (see Lemma 3.2 for review). We record [7, Theorem 0.3] as follows:

Theorem 1.3. *If $\mathcal{L}$ is an affine subspace of $\mathbb{R}^n$ and $\mathcal{M}$ is a nondegenerate submanifold in $\mathcal{L}$, then*

$$\omega(\mathcal{M}) = \omega(\mathcal{L}) = \inf \{\omega(\mathbf{x}) \mid \mathbf{x} \in \mathcal{L}\} = \inf \{\omega(\mathbf{x}) \mid \mathbf{x} \in \mathcal{M}\}. \tag{1.5}$$

In this paper we will be dealing with multiplicative version of the above concepts. We define

$$\Pi_+(\mathbf{y}) \overset{\mathbf{def}}{=} \prod_{i=1}^{n} |y_i|_+, \quad \text{where} \quad |y_i|_+ = \max(1, |y_i|), \tag{1.6}$$

$$\omega^\times(\mathbf{y}) = \sup \left\{ v \,\middle|\, \exists\infty \text{ many } \mathbf{q} \in \mathbb{Z}^n \text{ with } |\langle \mathbf{q}, \mathbf{y} \rangle + p| < \Pi_+(\mathbf{q})^{-v/n} \text{ for some } p \in \mathbb{Z} \right\}. \tag{1.7}$$

In the spirit of (1.3) we define multiplicative Diophantine exponents of manifolds and measures as

$$\omega^\times(\mathcal{M}) = \omega^\times(\mu) \overset{\mathbf{def}}{=} \sup\{v \mid \mu\{\mathbf{y} \mid \omega^\times(\mathbf{y}) > v\} > 0\}, \tag{1.8}$$

where μ is the measure class of Riemannian volume on $\mathcal{M}$.

From definitions we derive $\omega^\times(\mathbf{y}) \geq \omega(\mathbf{y})$. We call $\mathbf{y}$ very well multiplicatively approximable (VWMA) if $\omega^\times(\mathbf{y}) > n$. It can be proved that the set of VWMA vectors has zero Lebesgue measure. Following the terminology of [13], we call $\mathcal{M}$ strongly extremal if almost all $\mathbf{y} \in \mathcal{M}$ are not VWMA. Strong extremality implies extremality, and to prove a manifold to be strongly extremal is often more difficult to prove it to be just extremal.

A. Baker conjectured that the curve (1.4) is strongly extremal [1] in 1975. Proof of this conjecture was based on dynamical approach proposed in [9]. [9] also proved that nondegenerate manifolds of $\mathbb{R}^n$ are strongly extremal. In [6] D. Kleinbock gave a necessary and sufficient condition for an arbitrary affine subspace to be strongly extremal and showed that strong extremality of an affine space is inherited by its nondegenerate submanifolds. [6] also showed that a subspace is strongly extremal iff it contains at least one not VWMA vector. [2] gave a detailed account of historical

and recent development in the study of multiplicative Diophantine approximation, and in particular the renowned Littlewood's conjecture [2, §5].

This paper will calculate multiplicative Diophantine exponents of hyperplanes and their nondegenerate submanifolds. We follow the strategy of associating Diophantine property of vectors with behavior of certain trajectories in the space of lattices [9, 6]. In this process we will be considering multiparameter actions as opposed to one parameter ones which work well for standard Diophantine approximation problems. Combined with dynamics we use nondivergence estimates in its strengthened format [7] (see Lemma 3.2 of §3) to prove the following:

Theorem 1.4. *If $\mathcal{L}$ is a hyperplane of $\mathbb{R}^n$ and $\mathcal{M}$ is a nondegenerate submanifold in $\mathcal{L}$, then*

$$\omega^{\times}(\mathcal{L}) = \omega^{\times}(\mathcal{M}) = \inf\left\{\omega^{\times}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{L}\right\} = \inf\left\{\omega^{\times}(\mathbf{x}) \mid \mathbf{x} \in \mathcal{M}\right\}. \tag{1.9}$$

Theorem 1.4 shows that multiplicative Diophantine exponents of hyperplanes are inherited by their nondegenerate submanifolds. We will also calculate explicitly Diophantine exponents of such spaces in terms of the coefficients of their parameterizing maps. In §4 we will establish

Theorem 1.5. *Let $\mathcal{L}$ be a hyperplane of $\mathbb{R}^n$ defined by*

$$(x_1, x_2, \ldots, x_{n-1}) \to (a_1x_1 + a_2x_2 + \ldots + a_{n-1}x_{n-1} + a_n, x_1, x_2, \ldots, x_{n-1}). \tag{1.10}$$

Denote vector $(a_1, \ldots, a_{n-1}, a_n) \in \mathbb{R}^n$ by $\mathbf{a}$. Suppose that $s-1$ is equal to the number of nonzero elements in $\{a_1, \ldots, a_{n-1}\}$. Then we have

$$\omega^{\times}(\mathcal{L}) = \max\left(n, \frac{n}{s}\sigma(\mathbf{a})\right), \tag{1.11}$$

where

$$\sigma(\mathbf{a}) = \sup\left\{v \mid \exists\, \infty \text{ many } q \in \mathbb{Z} \text{ with } \|q\mathbf{a} + \mathbf{p}\| < |q|^{-v} \text{ for some } \mathbf{p} \in \mathbb{Z}^n\right\}. \tag{1.12}$$

From Theorem 1.5 we see that multiplicative Diophantine exponents of $\mathcal{L}$ and its nondegenerate submanifolds are dependent on the parameter s. Moreover s takes on integral values from 1 to n and is dependent on the first $n-1$ terms of $\mathbf{a}$ while unaffected by the last term a_n.

By comparison as a special case of [7, Theorem 0.2], for hyperplane $\mathcal{L}$ described by (1.10), we have

$$\omega(\mathcal{L}) = \max(n, \sigma(\mathbf{a})). \tag{1.13}$$

Consequently

$$\omega^{\times}(\mathcal{L}) = \omega(\mathcal{L}) \quad \text{iff} \quad s = n \quad \text{iff} \quad a_1a_2\cdots a_{n-1} \neq 0; \tag{1.14}$$

$$\omega^{\times}(\mathcal{L}) > \omega(\mathcal{L}) \quad \text{iff} \quad s < n \quad \text{iff} \quad a_1a_2\cdots a_{n-1} = 0. \tag{1.15}$$

In this way we exhibit classes of affine subspaces which are extremal but not strongly extremal. The main result of this paper is actually much more general than Theorem 1.4. We will be considering maps from Besicovitch metric spaces endowed with Federer measures (we postpone definitions of terminology till §3).

2. Dynamics

We will study homogeneous dynamics and how it relates to Diophantine approximation of vectors. First we define the space of unimodular lattices as follows:

$$\Omega_{n+1} \stackrel{\mathbf{def}}{=} \mathrm{SL}(n+1,\mathbb{R}) \big/ \mathrm{SL}(n+1,\mathbb{Z}). \tag{2.1}$$

Ω_{n+1} is noncompact, and can be decomposed as

$$\Omega_{n+1} = \bigcup_{\epsilon>0} K_\epsilon, \tag{2.2}$$

where

$$K_\epsilon = \Big\{ \Lambda \in \Omega_{n+1} \mid \|v\| \geq \epsilon \text{ for all nonzero } v \in \Lambda \Big\}. \tag{2.3}$$

Each K_ϵ is compact by Mahler's compactness criterion.

Remark 2.1. $\|\cdot\|$ *can be any norm on* $\mathbb{R}^{n+1}$ *and any two such norms are equivalent. We assume that it is the Euclidean norm from now on.*

We set

$$g_{\mathbf{t}} = \mathrm{diag}\Big\{ e^{-t_1}, \ldots, e^{-t_n}, e^t \Big\} \in \mathrm{SL}(n+1,\mathbb{R}), \tag{2.4}$$

where

$$t_i \geq 0, \quad t = \sum_{i=1}^n t_i, \quad \mathbf{t} = (t_1, \ldots, t_n). \tag{2.5}$$

Also set

$$u_{\mathbf{y}} = \begin{pmatrix} I_n & 0 \\ \mathbf{y} & 1 \end{pmatrix}. \tag{2.6}$$

The lattice $u_{\mathbf{y}}\mathbb{Z}^{n+1}$ takes on the form

$$u_{\mathbf{y}}\mathbb{Z}^{n+1} = \left\{ \left(\begin{array}{c} \mathbf{q} \\ \mathbf{q}\mathbf{y} + p \end{array} \right) \middle| \ \mathbf{q} \in \mathbb{Z}^n, \mathbf{p} \in \mathbb{Z} \right\}. \tag{2.7}$$

Also we define

$$W_v^\times \stackrel{\mathrm{def}}{=} \Big\{ \mathbf{y} \in \mathbb{R}^n \bigm| \omega^\times(\mathbf{y}) \geq v \Big\}. \tag{2.8}$$

By definition

$$\omega^\times(\mathbf{y}) = \sup \Big\{ v \bigm| \mathbf{y} \in W_v^\times \Big\}. \tag{2.9}$$

When we have $g_{\mathbf{t}}$ act on vectors in $u_{\mathbf{y}}\mathbb{Z}^{n+1}$ as defined by (2.7), the first n components will be contracted and the last one expanded. We propose the following lemma which shows a correlation between $\omega^\times(\mathbf{y})$ and trajectories of certain lattices in Ω_{n+1}. The original format stems from [6, Lemma 5.1], but what we need here is stronger and more precise.

Lemma 2.2. *Suppose we are given a positive integer* k $(1 \leq k \leq n)$ *and a subset* E *of* $\mathbb{R} \times \mathbb{Z}^{n+1}$ *which is discrete and homogeneous with respect to positive integers, and satisfies the condition that for every* $(x, \mathbf{z}) \in E$*, exactly* k *entries of* $\mathbf{z}$ *are nonzero. Take* $v > n$ *and* $c_k = \frac{v-n}{kv+n}$*, then the following are equivalent:*

(i) $\exists (x, \mathbf{z}) \in E$ *with arbitrarily large* $\|\mathbf{z}\|$ *such that*

$$|x| \leq \Pi_+(\mathbf{z})^{-v/n} \tag{2.10}$$

(ii) $\exists$ *an unbounded set of* $\mathbf{t} \in \mathbb{R}^n_+$ *such that for some* $(x, \mathbf{z}) \in E\backslash\{0\}$ *we have*

$$\max\left(e^t|x|, \quad e^{-t_i}|z_i|\right) \leq e^{-c_k t}, \quad 1 \leq i \leq n \tag{2.11}$$

Proof. Suppose (i) holds. Without loss of generality, assume $|z_i| \geq 1$ for $i \leq k$ and $z_i = 0$ for $i > k$. Define t by

$$e^{(1-kc_k)t} = \Pi_+(\mathbf{z}) = |z_1 \ldots z_k|. \tag{2.12}$$

Note that $c_k < 1/k$ from its definition $c_k = \frac{v-n}{kv+n}$, and t defined in the above equation is nonnegative thereof.

Then for every t define t_i by

$$e^{-t_i}|z_i| = e^{-c_k t} \quad \text{if} \quad 1 \leq i \leq k, \quad t_i = 0 \quad \text{if} \quad i > k. \tag{2.13}$$

Note that from (2.12) and (2.13) it is verified that $t = \sum_{i=1}^k t_i$. And we have

$$e^t|x| \leq e^t \Pi_+(\mathbf{z})^{-v/n} = e^t e^{(1-kc_k)(-v/n)t} = e^{t+(1-kc_k)(-v/n)t}. \tag{2.14}$$

Plugging in $c_k = \frac{v-n}{kv+n}$, we get

$$1 + (1 - kc_k)(-v/n) = -c_k. \tag{2.15}$$

Hence

$$e^{t+(1-kc_k)(-v/n)t} = e^{-c_k t}. \tag{2.16}$$

Hence (ii) is satisfied. In addition, by taking $\|\mathbf{z}\|$ arbitrarily large we produce arbitrarily large $\Pi_+(\mathbf{z})$ and t from (2.12).

Suppose (ii) holds. Because $(x, \mathbf{z}) \in E$ by reordering entries of $\mathbf{z}$ such that $|z_i| \geq 1$ for $i \leq k$ and $z_i = 0$ for $i > k$, we have

$$|z_i| \leq e^{t_i - c_k t} \quad \text{if} \quad i \leq k, \qquad |x| \leq e^{-(1+c_k)t}. \tag{2.17}$$

$$\Pi_+(\mathbf{z}) = |z_1 \ldots z_k| \leq e^{(t_1 - c_k t)+(t_2 - c_k t)+\ldots+(t_k - c_k t)} = e^{t_1+\ldots+t_k - kc_k t} \leq e^{t - kc_k t}. \tag{2.18}$$

By plugging in $c_k = \frac{v-n}{kv+n}$, we get

$$e^{-(1+c_k)t} = (e^{(1-kc_k)t})^{-v/n}. \tag{2.19}$$

Hence

$$|x| \leq e^{-(1+c_k)t} = (e^{(1-kc_k)t})^{-v/n} \leq \Pi_+(\mathbf{z})^{-v/n}. \tag{2.20}$$

Also by the discreteness of E, if $\|\mathbf{z}\|$ has a uniform bound while $|x|$ tends to zero, $(0, \mathbf{z}_0) \in E$ for some nonzero $\mathbf{z}_0$ and any integral multiple of $(0, \mathbf{z}_0)$ will satisfy (2.10). Obviously $\|p\mathbf{z}_0\|$ tends to infinity when the integer p tends to infinity. Therefore (i) is established. $\square$

Remark 2.3. *In* (2.11)*, because* $|z_i| \leq e^{t_i - c_k t}$*, we have* $t_i - c_k t \geq 0$ *for at least* k *values of* i*. This information is important because of the following elementary observation which plays an indispensable role in the proof of Lemma 4.5 in §4:*

Lemma 2.4. *Suppose* $p \in \mathbb{Z}$ *and* $|p| \leq e^\alpha$*. If* $\alpha \geq 0$ *then we have* $|p|_+ \leq e^\alpha$*.*

Proof. From (1.6) directly. $\square$

Remark 2.5. *If* $\alpha < 0$*, then* $|p| \leq e^\alpha$ *does not imply* $|p|_+ \leq e^\alpha$*. This distinction is important because in multiplicative Diophantine approximation we think of* $|p|_+$ *instead of* $|p|$*.*

We define

$$\mathbb{Z}_k^{n+1} = \left\{ (\mathbf{q}, p) = (q_1, \dots, q_n, p) \in \mathbb{Z}^{n+1} \middle| \text{ exactly } k \text{ entries of } \mathbf{q} \text{ are nonzero} \right\}. \tag{2.21}$$

Apparently

$$\mathbb{Z}^{n+1} = \bigcup_{k=0}^{n} \mathbb{Z}_k^{n+1}. \tag{2.22}$$

In light of Lemma 2.2, if we set $v > n$, $\mathbf{y} \in \mathbb{R}^n$ and $E = \left\{ (|\langle \mathbf{q}, \mathbf{y}\rangle + p|, \mathbf{q}) \middle| (\mathbf{q}, p) \in \mathbb{Z}_k^{n+1} \right\}$, condition (i) of Lemma 2.2 implies that

$$\mathbf{y} \in W_v^{\times}. \tag{2.23}$$

Condition (ii) becomes equivalent to: $\exists$ an unbounded set of $\mathbf{t} \in \mathbb{R}_+^n$ such that

$$t_i \geq c_k t \text{ for at least } k \text{ values of } i, \tag{2.24}$$

and

$$g_{\mathbf{t}} u_{\mathbf{y}} \mathbb{Z}_k^{n+1} \text{ contains at least one vector with norm } \leq e^{-c_k t}. \tag{2.25}$$

Furthermore

$$c_k = \frac{v-n}{kv+n} \Longleftrightarrow v = \frac{n + nc_k}{1 - kc_k}, \qquad 1 \leq k \leq n \tag{2.26}$$

Recall that by (2.5) $\mathbf{t}$ is multiparameter vector in $\mathbb{R}_+^n$ and $t = \sum_{i=1}^n t_i$. If we set

$$\gamma_k(\mathbf{y}) = \sup \Big\{ c_k \mid (2.25) \text{ holds for an unbounded set of } \mathbf{t} \in \mathbb{R}_+^n \text{ satisfying } (2.24) \Big\}, \tag{2.27}$$

we have the following theorem, which is the main result of this section.

Theorem 2.6. *$\forall \mathbf{y} \in \mathbb{R}^n$, we have*

$$\omega^{\times}(\mathbf{y}) = \max_{1 \leq k \leq n} \frac{n + n\gamma_k(\mathbf{y})}{1 - k\gamma_k(\mathbf{y})}. \tag{2.28}$$

Proof. We first prove that in (2.27) we can have $\mathbf{t} \in \mathbb{Z}_+^n$ as opposed to $\mathbf{t} \in \mathbb{R}_+^n$. We adopt arguments of [9, Corollary 2.2] here. Suppose for some $\mathbf{t} = (t_1, t_2, \dots, t_n)$ and $t = \sum t_i$, we have $t_i \geq c_k t$ for at least k values of i as well as

$$g_{\mathbf{t}} u_{\mathbf{y}} \mathbb{Z}_k^{n+1} \text{ contains at least one vector with norm } \leq e^{-c_k t}.$$

Denote by $[\mathbf{t}]$ the vector consisting of integer parts of t_i. Then the ratio of lengths of the shortest vector of $g_{[\mathbf{t}]} u_{\mathbf{y}} \mathbb{Z}_k^{n+1}$ and the shortest vector of $g_{\mathbf{t}} u_{\mathbf{y}} \mathbb{Z}_k^{n+1}$ is bounded from above by

$$\left\| g_{\mathbf{t}} g_{[\mathbf{t}]}^{-1} \right\| = \left\| g_{\mathbf{t} - [\mathbf{t}]} \right\| \leq e^n.$$

Hence we get

$$g_{[\mathbf{t}]} u_{\mathbf{y}} \mathbb{Z}_k^{n+1} \text{ contains at least one vector with norm } \leq e^n e^{-c_k t}.$$

When t is large we can decrease c_k slightly to c_k' and get

$$g_{[\mathbf{t}]} u_{\mathbf{y}} \mathbb{Z}_k^{n+1} \text{ contains at least one vector with norm } \leq e^{-c_k' t}$$

as well as

$$[t_i] \geq c_k' \left(\sum [t_i] \right) \text{ for at least } k \text{ values of } i.$$

Therefore

$$\gamma_k(\mathbf{y}) = \sup \Big\{ c_k \mid (2.25) \text{ holds for an unbounded set of } \mathbf{t} \in \mathbb{Z}_+^n \text{ satisfying } (2.24) \Big\}.$$

Next we show

$$\omega^{\times}(\mathbf{y}) \geq \frac{n + n\gamma_k(\mathbf{y})}{1 - k\gamma_k(\mathbf{y})}, \quad 1 \leq k \leq n.$$

To see this, apply Lemma 2.2 n times, letting k go from 1 to n. For each k, condition (ii) of Lemma 2.2 implies condition (i), which in turn implies that $\mathbf{y} \in W_v^{\times}$ or $\omega^{\times}(\mathbf{y}) \geq v$.

On the other hand, (2.23) clearly forces condition (i) of Lemma 2.2 to hold for some k between 1 and n. Hence

$$\omega^{\times}(\mathbf{y}) \leq \max_{1 \leq k \leq n} \frac{n + n\gamma_k(\mathbf{y})}{1 - k\gamma_k(\mathbf{y})}.$$

(2.28) is therefore established. □

Suppose ν is a measure on $\mathbb{R}^n$ and $v > n$, by definition

$$\omega^{\times}(\nu) \leq v \quad \text{if and only if} \quad \nu(W_u^{\times}) = 0, \quad \forall u > v. \tag{2.29}$$

By the Borel-Cantelli Lemma and the above theorem, a sufficient condition for $\omega^{\times}(\nu) \leq v$ is:

Condition 2.7. *$\forall k$ $(1 \leq k \leq n)$, $\forall d_k > c_k$, we have*

$$\sum_{\substack{\mathbf{t} \in \mathbb{Z}_+^n, \\ t_i \geq d_k t \text{ for at least} \\ k \text{ values of } i}} \nu\Big(\Big\{\mathbf{y} \,\big|\, g_{\mathbf{t}} u_{\mathbf{y}} \mathbb{Z}_k^{n+1} \text{ has at least one nonzero vector with norm} \leq e^{-d_k t}\Big\}\Big) < \infty. \tag{2.30}$$

Remark 2.8. *Condition 2.7 is helpful because it allows us to find upperbounds of $\omega^{\times}(\lambda)$ by applying quantitative nondivergence in the next section. The restriction similar to* (2.24) *will be used in the proof of Lemma 4.5 in §4.*

3. Quantitative Nondivergence

Before stating nondivergence quantitative results, we first introduce an assembly of relevant concepts developed in [7], [8] and [9]. A metric space X is called $N-$ *Besicovitch* if for any bounded subset A and any family β of nonempty open balls of X such that each $x \in A$ is a center of some ball of β, there is a finite or countable subfamily $\{\beta_i\}$ of β covering A with multiplicity at most N. X is *Besicovitch* if it is $N-$ *Besicovitch* for some N.

Let μ be a locally finite Borel measure on X, U an open subset of X with $\mu(U) > 0$. Following [8] we call μ $D-$ *Federer* on U if

$$\sup_{\substack{x \in \operatorname{supp} \mu,\, r>0 \\ B(x,3r) \subset U}} \frac{\mu(B(x,3r))}{\mu(B(x,r))} < D \tag{3.1}$$

μ is said to be *Federer* if for μ-a.e. $x \in X$ there exists a neighborhood U of x and $D > 0$ such that μ is $D-$ *Federer* on U.

An important illustration of the above notions is that $\mathbb{R}^d$ is Besicovitch and λ, the Lebesgue measure is *Federer*. Many natural measures supported on fractals are also known to be *Federer* (see [8] for technical details).

For a subset B of X and a function f from B to a normed space with norm $\|\ \|$, we define $\|f\|_B = \sup_{x \in B} \|f(x)\|$. If μ is a Borel measure on X and B a subset of X with $\mu(B) > 0$ $\|f\|_{\mu,B}$ is set to be $\|f\|_{B \cap \operatorname{supp}\ \mu}$.

A function $f : X \to \mathbb{R}$ is called (C,α)-good on $U \subset X$ with respect to μ if for any open ball B centered in supp μ one has

$$\forall \varepsilon > 0 \qquad \mu(\{x \in B \mid |f(x)| < \varepsilon\}) \le C \left(\frac{\varepsilon}{\|f\|_{\mu,B}}\right)^{\alpha} \mu(B). \tag{3.2}$$

Roughly speaking a function is (C,α)-good if the set of points where it takes small value has small measure. In Lemma 3.2 we use the fact that functions of the form $\mathbf{x} \to \|h(\mathbf{x})\Gamma\|$, where Γ runs through subgroups of $\mathbb{Z}^{n+1}$, are (C,α)-good with uniform C and α.

Let $\mathbf{f} = (f_1, \dots, f_n)$ be a map from X to $\mathbb{R}^n$. Following [7] we say that $(\mathbf{f},\mu)$ is good at $x \in X$ if there exists a neighborhood V of x such that any linear combination of $1, f_1, \dots, f_n$ is (C,α)-good on V with respect to μ and $(\mathbf{f},\mu)$ is good if $(\mathbf{f},\mu)$ is good at μ-almost every point. Reference to measure will be omitted if $\mu = \lambda$, and we will simply say that $\mathbf{f}$ is good or good at x. For example polynomial maps are good. [6] proved the following result:

Lemma 3.1. *Let $\mathcal{L}$ be an affine subspace of $\mathbb{R}^n$, and let $\mathbf{f}$ be a smooth map from U, an open subset of $\mathbb{R}^d$ to $\mathcal{L}$ which is nondegenerate at $\mathbf{x} \in U$; then $\mathbf{f}$ is good at $\mathbf{x}$.*

Furthermore if $\mathcal{L}$ is an affine subspace of $\mathbb{R}^n$ and $\mathbf{f}$ a map from X into $\mathcal{L}$, following [7] we say $(\mathbf{f},\mu)$ is nonplanar in $\mathcal{L}$ at $x \in$ supp μ if $\mathcal{L}$ is equal to the intersection of all affine subspaces containing $\mathbf{f}(B \cap \text{supp } \mu)$ for any open neighborhood B of x. $(\mathbf{f},\mu)$ is nonplanar in $\mathcal{L}$ if $(\mathbf{f},\mu)$ is nonplanar in $\mathcal{L}$ at μ-a.e. x. We skip saying μ when $\mu = \lambda$ and skip $\mathcal{L}$ if $\mathcal{L} = \mathbb{R}^n$. From definition $(\mathbf{f},\mu)$ is nonplanar if and only if for any open B of positive measure, the restrictions of $1, f_1, \dots, f_n$ to $B \cap \text{supp } \mu$ are linearly independent over $\mathbb{R}$. Clearly nondegeneracy in $\mathcal{L}$ implies nonplanarity in $\mathcal{L}$. Nondegenerate smooth maps from $\mathbb{R}^d$ to $\mathbb{R}^n$ as in Lemma 3.1 give typical examples of nonplanarity.

Let Γ be any discrete subgroup of $\mathbb{R}^k$ we denote by $rk(\Gamma)$ the rank of Γ when viewed as a $\mathbb{Z}$-module. The following is exactly [7, Theorem 2.2].

Lemma 3.2. *Let m, $N \in \mathbb{N}$ and $C, D, \alpha, \rho > 0$ and suppose we are given an $N-$ Besicovitch metric space X, a ball $B = B(x_0, r_0) \subset X$, a measure μ which is $D-$ Federer on $\tilde{B} = B(x_0, 3^m r_0)$ and a map $h\colon \tilde{B} \to \mathrm{GL}_m(\mathbb{R})$. Assume the following two conditions hold:*

(i) $\forall\, \Gamma \subset \mathbb{Z}^m$, *the function $x \to \|h(x)\Gamma\|$ is (C,α)-good on $\tilde{B}$ with respect to μ;*
(ii) $\forall\, \Gamma \subset \mathbb{Z}^m$, $\|h(\cdot)\Gamma\|_{\mu,B} \ge \rho^{rk(\Gamma)}$.

Then for any positive $\epsilon \le \rho$, we have

$$\mu\Big(\Big\{x \in B \mid h(x)\mathbb{Z}^m \notin K_\epsilon\Big\}\Big) \le mC(ND^2)^m \left(\frac{\epsilon}{\rho}\right)^{\alpha} \mu(B). \tag{3.3}$$

Proposition 3.3. *Let X be a Besicovitch metric space, $B = B(\mathbf{x}, r) \subset X$, μ a measure which is $D-$ Federer on $\tilde{B} = B(\mathbf{x}, 3^{n+1}r)$ for some $D > 0$ and $\mathbf{f}$ a continuous map from $\tilde{B}$ to $\mathbb{R}^n$. Given $v \ge n$, let $c_k = \frac{v-n}{kv+n}$ where $1 \le k \le n$ and assume that*

(i) $\exists C, \alpha > 0$ *such that all the functions $\mathbf{x} \to \left\|g_{\mathbf{t}} u_{\mathbf{f}(\mathbf{x})} \Gamma\right\|$, $\Gamma \subset \mathbb{Z}^{n+1}$ are (C,α)-good on $\tilde{B}$ with respect to μ*
(ii) $\forall k$ $(1 \le k \le n)$, $\forall\, d_k > c_k$, $\exists T = T(d_k) > 0$ *such that for any vector $\mathbf{t} \in \mathbb{Z}^n_+$ with $t \ge T$ and $t_i \ge d_k t$ for at least k values of i and any $\Gamma \subset \mathbb{Z}^{n+1}$,*

we have

$$\left\| g_{\mathbf{t}} u_{\mathbf{f}(\cdot)} \Gamma \right\|_{\mu,B} \geq e^{-rk(\Gamma) d_k t}. \tag{3.4}$$

Then $\omega^{\times}(\mathbf{f}_*(\mu|_B)) \leq v$.

Proof. Apply Lemma 3.2 n times, letting k go from 1 to n. For each iteration set $m = n+1$ and $\nu = \mathbf{f}_*(\mu|_B)$.

$\forall k$, $\forall d_k > c_k$ and for all $\mathbf{t} \in \mathbb{Z}_+^n$ satisfying the condition that $t_i \geq d_k t$ for at least k values of i, set $h_k(\mathbf{x}) = g_{\mathbf{t}} u_{\mathbf{f}(\mathbf{x})}$. We see that condition (i) of Lemma 3.2 agrees with condition (i) of Proposition 3.3.

For the other condition, set $\rho_k^t = e^{-\frac{c_k + d_k}{2} t}$ and $\epsilon_k^t = e^{-d_k t}$. Note that

$$d_k > c_k \Leftrightarrow \epsilon_k^t < \rho_k^t.$$

Also we have

$$\frac{\epsilon_k^t}{\rho_k^t} = e^{-\frac{d_k - c_k}{2} t}.$$

It follows that condition (ii) of Proposition 3.3 implies condition (ii) of Lemma 3.2 for $t > T(\frac{c_k + d_k}{2})$. Hence by Lemma 3.2, for any fixed $\mathbf{t} \in \mathbb{Z}_+^n$ with $t \geq T$ and $t_i \geq d_k t$ for at least k values of i, we have

$$\begin{aligned} \nu\left(\left\{\mathbf{y} \mid g_{\mathbf{t}} u_{\mathbf{y}} \mathbb{Z}^{n+1} \notin K_{e^{-d_k t}}\right\}\right) &= \mu\left(\left\{\mathbf{x} \in B \mid h_k(\mathbf{x}) \mathbb{Z}^{n+1} \notin K_{e^{-d_k t}}\right\}\right) \\ &\leq const \cdot e^{-\alpha \frac{d_k - c_k}{2} t} \mu(B). \end{aligned} \tag{3.5}$$

We have the obvious identity

$$\sum_{\mathbf{t} \in \mathbb{Z}_+^n}^{\infty} \nu\Big(\left\{\mathbf{y} \mid g_{\mathbf{t}} u_{\mathbf{y}} \mathbb{Z}^{n+1} \notin K_{e^{-d_k t}}\right\} \Big) = \sum_{l=1}^{\infty} \sum_{\mathbf{t} \in \mathbb{Z}_+^n,\ t = l} \nu\Big(\left\{\mathbf{y} \mid g_{\mathbf{t}} u_{\mathbf{y}} \mathbb{Z}^{n+1} \notin K_{e^{-d_k l}}\right\} \Big). \tag{3.6}$$

Since for each $l \in \mathbb{N}$, the possible number of $\mathbf{t} \in \mathbb{Z}_+^n$ with $t = l$ is bounded from above by $(l+1)^n$, we get from (3.6)

$$\sum_{\substack{\mathbf{t} \in \mathbb{Z}_+^n, \\ t_i \geq d_k t \text{ for at least} \\ k \text{ values of } i}} \nu\Big(\left\{\mathbf{y} \mid g_{\mathbf{t}} u_{\mathbf{y}} \mathbb{Z}^{n+1} \notin K_{e^{-d_k t}}\right\} \Big) \leq \sum_{l=1}^{\infty} (l+1)^n const \cdot e^{-\alpha \frac{d_k - c_k}{2} l} \mu(B) < \infty. \tag{3.7}$$

Since $h_k(\mathbf{x}) \mathbb{Z}_k^{n+1} \subset h_k(\mathbf{x}) \mathbb{Z}^{n+1}$, we have

$$\begin{aligned} &\left\{\mathbf{x} \in B \mid h_k(\mathbf{x}) \mathbb{Z}_k^{n+1} \text{ has at least one vector with norm} \leq e^{-d_k t}\right\} \\ &\subset \left\{\mathbf{x} \in B \mid h_k(\mathbf{x}) \mathbb{Z}^{n+1} \notin K_{e^{-d_k t}}\right\}. \end{aligned} \tag{3.8}$$

Moreover we note that the restriction $t_i \geq d_k t$ for at least k values of i is also present in Condition 2.7. We let k range over all integers between 1 and n and Condition 2.7 is satisfied. $\square$

4. Proof of Main Theorems

To prove the theorems, we first calculate $\|g_{\mathbf{t}} u_{\mathbf{f}(\cdot)} \Gamma\|_{\mu,B}$ in (3.4). The following exterior algebraic computation comes from [7] and [9].

Suppose $\mathbb{R}^{n+1}$ has standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_{n+1}$, and if we extend the Euclidean structure of $\mathbb{R}^{n+1}$ to $\bigwedge^j(\mathbb{R}^{n+1})$, then for index sets

$$I = \left\{i_1, i_2, \ldots, i_j\right\} \subset \left\{1, 2, \ldots, n+1\right\}, \quad i_1 < i_2 < \cdots < i_j \tag{4.1}$$

$\{\mathbf{e}_I \mid \mathbf{e}_I = \mathbf{e}_{i_1} \wedge \mathbf{e}_{i_2} \wedge \cdots \wedge \mathbf{e}_{i_j}, \quad \#I = j\}$ form an orthogonal basis of $\bigwedge^j(\mathbb{R}^{n+1})$ when I range over all index sets of the form (4.1). If a discrete subgroup $\Gamma \subset \mathbb{R}^{n+1}$ of rank j is viewed as a $\mathbb{Z}$-module with basis $\mathbf{v}_1, \ldots, \mathbf{v}_j$, then we may represent it by exterior product $\mathbf{w} = \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_j$. Observing $\|\Gamma\| = \|\mathbf{w}\|$, we will be able to compute $\|g_t u_{\mathbf{f}} \Gamma\|_{\mu,B}$ as in (3.4) directly.

We assume from now on that J and I stand for index sets: J is of order $j-1$ and I is of order j. Given $\mathbf{y} = (y_1, \ldots, y_n)$, we set $y_{n+1} = 1$ and get $u_{\mathbf{y}}$ as in (2.6). We get

$$u_{\mathbf{y}} \mathbf{e}_I = \mathbf{e}_I, \qquad \text{if } n+1 \in I;$$

$$u_{\mathbf{y}} \mathbf{e}_I = \mathbf{e}_I \pm \sum_{i \in I} y_i \mathbf{e}_{I \cup \{n+1\} \setminus \{i\}} \qquad \text{otherwise.} \tag{4.2}$$

Hence

$$u_{\mathbf{y}} \mathbf{w} = \sum_{I \subset \{1,\ldots,n\}} \pm \langle \mathbf{e}_I, \mathbf{w} \rangle \mathbf{e}_I + \sum_{J \subset \{1,\ldots,n\}} \left(\sum_{i=1}^{n+1} \pm \langle \mathbf{e}_i \wedge \mathbf{e}_J, \mathbf{w} \rangle y_i \right) \mathbf{e}_J \wedge \mathbf{e}_{n+1}. \tag{4.3}$$

Since $g_{\mathbf{t}} = \operatorname{diag}\{e^{-t_1}, \ldots, e^{-t_n}, e^t\}$, we have

$$g_{\mathbf{t}} \mathbf{e}_i = e^{-t_i} \mathbf{e}_i \quad (1 \le i \le n); \tag{4.4}$$

$$g_{\mathbf{t}} \mathbf{e}_{n+1} = e^t \mathbf{e}_{n+1}; \tag{4.5}$$

$$\begin{aligned} g_{\mathbf{t}} u_{\mathbf{y}} \mathbf{w} = & \sum_{I \subset \{1,\ldots,n\}} e^{-\sum_{i \in I} t_i} \pm \langle \mathbf{e}_I, \mathbf{w} \rangle \mathbf{e}_I \\ & + \sum_{J \subset \{1,\ldots,n\}} e^{t - \sum_{i \in J} t_i} \left(\sum_{i=1}^{n+1} \pm \langle \mathbf{e}_i \wedge \mathbf{e}_J, \mathbf{w} \rangle y_i \right) \mathbf{e}_J \wedge \mathbf{e}_{n+1}. \end{aligned} \tag{4.6}$$

For $\mathbf{f} = (f_1, f_2, \ldots, f_n) : \tilde{B} \to \mathbb{R}^n$ in (3.4), we set $f_{n+1} = 1$ and

$$\widetilde{\mathbf{f}} = (f_1, \ldots, f_n, 1).$$

Also set

$$\mathbf{c}(\mathbf{w})_i = \sum_{\substack{J \subset \{1,\ldots,n\} \\ \#J = j-1}} \pm \langle \mathbf{e}_i \wedge \mathbf{e}_J, \mathbf{w} \rangle \mathbf{e}_J \in \bigwedge^{j-1}(\mathbb{R}^{n+1}), \quad 1 \le i \le n+1, \tag{4.7}$$

$$\mathbf{c}(\mathbf{w}) = \begin{pmatrix} \mathbf{c}(\mathbf{w})_1 \\ \mathbf{c}(\mathbf{w})_2 \\ \vdots \\ \mathbf{c}(\mathbf{w})_{n+1} \end{pmatrix}. \tag{4.8}$$

Noting that $\mathbf{e}_I$ and $\mathbf{e}_J \wedge \mathbf{e}_{n+1}$ appearing in (4.6) are orthogonal, we have, up to some constant dependent on n only

(4.9)

$$\begin{aligned} \left\| g_{\mathbf{t}} u_{\mathbf{f}(\cdot)} \mathbf{w} \right\|_{\mu,B} & \asymp \max \left(e^{-\sum_{i \in I} t_i} \| \langle \mathbf{e}_I, \mathbf{w} \rangle \|, \quad e^{t - \sum_{i \in J} t_i} \left\| \sum_{i=1}^{n+1} \pm \langle \mathbf{e}_i \wedge \mathbf{e}_J, \mathbf{w} \rangle f_i \right\|_{\mu,B} \right) \\ & = \max \left(e^{-\sum_{i \in I} t_i} \| \langle \mathbf{e}_I, \mathbf{w} \rangle \|, \quad e^{t - \sum_{i \in J} t_i} \left\| \widetilde{\mathbf{f}}(\cdot) \mathbf{c}(\mathbf{w}) \right\|_{\mu,B} \right), \end{aligned}$$

where the maximum is taken over all index sets $I \subset \{1, \dots, n\}$ and $J \subset \{1, \dots, n\}$.

Following arguments of [7], we see that the value of $\|g_{\mathbf{t}} u_{\mathbf{f}(\cdot)} \mathbf{w}\|_{\mu,B}$ as in (4.9) is affected by the linear dependence relations between the components of $\widetilde{\mathbf{f}}$. We denote by $\mathcal{F}_{\mu,B}$ the $\mathbb{R}$-linear span of the restrictions of $f_1, \dots, f_n, 1$ to $B \cap \operatorname{supp} \mu$, denote its dimension by $l+1$, and choose functions $g_1, \dots, g_l : B \cap \operatorname{supp} \mu \to \mathbb{R}$ such that $g_1, \dots, g_l, 1$ form a basis of $\mathcal{F}_{\mu,B}$. This choice defines a matrix

$$R = (r_{i,j})_{\substack{i=1,\dots,l+1 \\ j=1,\dots,n+1}} \in M_{l+1,n+1} \tag{4.10}$$

formed by coefficients in the expansion of $f_1, \dots, f_n, 1$ as linear combinations of $g_1, \dots, g_l, 1$. In other words, with the notation $\widetilde{\mathbf{g}} = (g_1, \dots, g_l, 1)$, we have

$$\widetilde{\mathbf{f}}(\mathbf{x}) = \widetilde{\mathbf{g}}(\mathbf{x}) R, \quad \forall \mathbf{x} \in B \cap \operatorname{supp} \mu. \tag{4.11}$$

Therefore $\|\widetilde{\mathbf{f}}(\cdot)\mathbf{c}(\mathbf{w})\|_{\mu,B}$ can be replaced by $\|\widetilde{\mathbf{g}}(\cdot)R\mathbf{c}(\mathbf{w})\|_{\mu,B}$ and the latter, in view of linear independence of the components of $\widetilde{\mathbf{g}}$, simply by the norms of vectors $R\mathbf{c}(\mathbf{w})$ (up to some constant uniform in $\mathbf{w}$ yet dependent on $\mathbf{f}$, $\widetilde{\mathbf{g}}$, μ and B). The second assumption of Proposition 3.3 can be rewritten as:

Condition 4.1. $\forall k$ $(1 \le k \le n)$ $\quad \forall d_k > c_k$, $\exists T = T(d_k) > 0$ *such that for all* $\mathbf{t} \in \mathbb{Z}_+^n$ *with* $t \ge T$ *and* $t_i \ge d_k t$ *for at least* k *values of* i*, we have* $\forall j$ $(1 \le j \le n)$, $\forall$ *nonzero* $\mathbf{w} \in \bigwedge^j(\mathbb{Z}^{n+1})$,

$$\max\left(e^{-\sum_{i\in I} t_i}\|\langle \mathbf{e}_I, \mathbf{w}\rangle\|, \quad e^{t-\sum_{i\in J} t_i}\|R\mathbf{c}(\mathbf{w})\|\right) \ge e^{-j d_k t},$$

where the maximum is taken over all index sets $I \subset \{1, \dots, n\}$ *with order* j *and* $J \subset \{1, \dots, n\}$ *with order* $j-1$*. Matrix* R *is defined via* (4.11).

Remark 4.2. *k and j are independent variables: k arises from Lemma 2.2 while j is the rank of $\mathbf{w}$. R depends on the measure μ, the ball B, the map $\mathbf{f}$ as well as the choice of $\widetilde{\mathbf{g}}$.*

According to [7], the only way the ball B, the measure μ and the map $\mathbf{f}$ enter the above conditions is via the matrix R, which depends on B, μ and $\mathbf{f}$ and is not uniquely determined. However another choice of R would yield a condition equivalent to Condition 4.1. Let μ be a Federer measure on a Besicovitch metric space X and $\mathcal{L}$ a hyperplane of $\mathbb{R}^n$. We assume from now on that $\mathbf{f} \colon X \to \mathcal{L}$ is a continuous map such that $(\mathbf{f}, \mu)$ is nonplanar in $\mathcal{L}$. For a subset M of $\mathbb{R}^n$, define its affine span $\langle M \rangle_a$ to be the intersection of all affine subspaces of $\mathbb{R}^n$ containing M. By definition [7, §1], $(\mathbf{f}, \mu)$ is nonplanar in $\mathcal{L}$ iff

$$\mathcal{L} = \langle \mathbf{f}(B \cap \operatorname{supp} \mu)\rangle_a, \quad \forall \text{ open } B \subset X \text{ with } \mu(B) > 0. \tag{4.12}$$

Suppose

$$\mathbf{h} \colon \mathbb{R}^{n-1} \to \mathcal{L} = \langle \mathbf{f}(B \cap \operatorname{supp} \mu)\rangle_a \text{ is an affine isomorphism, and}$$

$$\widetilde{\mathbf{h}}(\mathbf{x}) = \widetilde{\mathbf{x}} R, \quad \mathbf{x} \in \mathbb{R}^{n-1}, \tag{4.13}$$

where $\widetilde{\mathbf{h}} = (h_1, \dots, h_n, 1)$ and $\widetilde{\mathbf{x}} = (x_1, \dots, x_{n-1}, 1)$. Then R and $\mathbf{g} = \mathbf{h}^{-1} \circ \mathbf{f}$ satisfy (4.11). $g_1, \dots, g_{n-1}, 1$ generate $\mathcal{F}_{\mu,B}$ and are linearly independent over $\mathbb{R}$. This way Condition 4.1 or the second assumption of Proposition 3.3 becomes a property of the space $\langle \mathbf{f}(B \cap \operatorname{supp} \mu)\rangle_a$ or $\mathcal{L}$. We can thus choose R uniformly for all measures μ, balls B and maps $\mathbf{f}$. Since the statement that Condition 4.1 holds for any R satisfying (4.11) is equivalent to the statement that it holds for some R

satisfying (4.11), we will make the most natural choices for $\mathcal{L}$ as described in (1.10): $X = \mathbb{R}^{n-1}$, $\mu = \lambda$ and the following map according to (4.13):

$$\widetilde{\mathbf{h}}(\mathbf{x}) = (h_1, \dots, h_n, 1)(\mathbf{x}) = (x_1, x_2, \dots, x_{n-1}, 1)R_0, \tag{4.14}$$

where R_0 is an $n \times (n+1)$ matrix defined by

$$R_0 = \begin{pmatrix} a_1 & \\ \vdots & I_n \\ a_{n-1} & \\ a_n & \end{pmatrix}. \tag{4.15}$$

We can replace an arbitrary R in Condition 4.1 by R_0 defined in (4.15) as long as $(\mathbf{f}, \mu)$ is nonplanar in $\mathcal{L}$.

Noting (4.9) and the fact that $e^{t-\sum_{i\in J} t_i} \geq 1$, we get

$$\left\| g_t u_{\mathbf{f}(\cdot)} \mathbf{w} \right\|_{\mu,B} \succ \| R_0 \mathbf{c}(\mathbf{w}) \|, \tag{4.16}$$

where $\succ$ implies some constant dependent on μ, B and $\mathbf{f}$.

Next we restate and reprove [6, Lemma 4.6].

Lemma 4.3. *For R_0 defined in (4.15) and nonzero $\mathbf{w} \in \bigwedge^j(\mathbb{Z}^{n+1})$, we have*

$$\| R_0 \mathbf{c}(\mathbf{w}) \| \geq 1 \text{ if } j > 1. \tag{4.17}$$

Proof. Suppose for some index set $I_1 = \{i_1, i_2, \dots, i_j\}$ we have $a = \langle \mathbf{e}_{I_1}, \mathbf{w} \rangle \in \mathbb{Z}$ and $a \neq 0$. Since $j > 1$, without loss of generality, we assume that $i_1 = 1$ and $i_2 = 2$. We consider the first entry of $\|R_0 \mathbf{c}(\mathbf{w})\| = \|a_1 \mathbf{c}(\mathbf{w})_1 + \mathbf{c}(\mathbf{w})_2\|$ and prove that $\|a_1 \mathbf{c}(\mathbf{w})_1 + \mathbf{c}(\mathbf{w})_2\| \geq 1$. Once this is proved the lemma will be established. Set $J_1 = \{2, i_3, , \dots, i_j\}$. Then $\mathbf{c}(\mathbf{w})_1$ has no term containing $\mathbf{e}_{J_1}$ because otherwise, by (4.7) we will have $1 \in J_1$. In other words, $\mathbf{c}(\mathbf{w})_1$ only has terms orthogonal to $\mathbf{e}_{J_1}$. In addition, $\mathbf{c}(\mathbf{w})_2 = \pm a \mathbf{e}_{J_1} +$ terms orthogonal to $\mathbf{e}_{J_1}$. Hence

$$\|a_1 \mathbf{c}(\mathbf{w})_1 + \mathbf{c}(\mathbf{w})_2\| \geq \| \pm a \mathbf{e}_{J_1} \| = |a| \geq 1. \tag{4.18}$$

□

Hence the assumptions of Proposition 3.3 are automatically fulfilled for subgroups Γ represented by $\mathbf{w}$ as above, because from (4.16) and (4.17) we get

$$\left\| g_t u_{\mathbf{f}(\cdot)} \mathbf{w} \right\|_{\mu,B} \succ 1 \text{ if } j > 1, \tag{4.19}$$

where $\succ$ implies some constant dependent on μ, B and $\mathbf{f}$.

Thus the second assumption of Proposition 3.3 or Condition 4.1 can be rewritten as:

Condition 4.4. $\forall k$ $(1 \leq k \leq n)$, $\forall d_k > c_k$, $\exists T = T(d_k) > 0$ *such that for any* $t \geq T$ *with* $t_i \geq d_k t$ *for at least* k *values of* i, $\forall$ *nonzero* $\mathbf{w} \in \mathbb{Z}^{n+1}$ *, we have*

$$\max\left(e^{-t_i} \left\| \langle \mathbf{e}_i, \mathbf{w} \rangle \right\|, \quad e^t \left\| R_0 \mathbf{c}(\mathbf{w}) \right\| \right) \geq e^{-d_k t}, \quad 1 \leq i \leq n, \tag{4.20}$$

Matrix R_0 is defined via (4.15).

In summary, when $\mathbf{f} \colon X \to \mathcal{L}$ is a continuous map such that $(\mathbf{f}, \mu)$ is nonplanar in $\mathcal{L}$, the second assumption of Proposition 3.3 $\Leftrightarrow$ Condition 4.1 $\Leftrightarrow$ Condition 4.4. The next lemma gives an account of what happens if the above conditions fail to hold.

Lemma 4.5. *Let μ be a Federer measure on a ball $B \subset X$, take $v > n$ and $c_k = \frac{v-n}{kv+n}$ $(1 \le k \le n)$. Let $\mathbf{f}$ be a continuous map from X to $\mathcal{L}$ such that $(\mathbf{f}, \mu)$ is nonplanar in $\mathcal{L}$ and the second assumption of Proposition 3.3 does not hold. Then $\mathbf{f}(B \cap \text{ supp } \mu) \subset W_u^\times$ for some $u > v$.*

Proof. If the second assumption of Proposition 3.3 or equivalently Condition 4.4 does not hold, $\exists k$ with $1 \le k \le n$, a sequence $t^j \to \infty$ and a sequence of nonzero integer vectors $\mathbf{w}^j$ such that for some $d_k > c_k$, we have $\forall \mathbf{x} \in B \cap$ supp μ

(4.21)
$$\left\|g_{\mathbf{t}^j} u_{\mathbf{f}(\mathbf{x})} \mathbf{w}^j\right\| \le e^{-d_k t^j}, \text{ where } t^j = \sum_{i=1}^n t_i^j \text{ and } t_i^j \ge d_k t^j \text{ for at least } k \text{ values of } i.$$

Equivalently, $\forall \mathbf{x} \in B \cap$ supp μ, $\exists m$ independent of k with $1 \le m \le n$, such that for an infinite subsequence of j, there exists nonzero vector v^j such that

$$\|v^j\| \le e^{-d_k t^j}, \quad v^j \in g_{\mathbf{t}^j} u_{\mathbf{f}(\mathbf{x})} \mathbb{Z}_m^{n+1}. \tag{4.22}$$

Recall that by (2.21)
(4.23)
$$\mathbb{Z}_m^{n+1} = \left\{ (\mathbf{q}, p) = (q_1, \dots, q_n, p) \in \mathbb{Z}^{n+1} \middle| \text{ exactly } m \text{ entries of } \mathbf{q} \text{ are nonzero} \right\}.$$

Consequently

$$\gamma_m(\mathbf{f}(\mathbf{x})) \ge d_k \tag{4.24}$$

We get from (2.28) that

$$\omega^\times(\mathbf{f}(\mathbf{x})) \ge \frac{n + nd_k}{1 - md_k} \tag{4.25}$$

If $m \ge k$, then because the function $a(x) = \frac{n+nd_k}{1-xd_k}$ increases as x increases, we get

$$\omega^\times(\mathbf{f}(\mathbf{x})) \ge \frac{n + nd_k}{1 - md_k} \ge \frac{n + nd_k}{1 - kd_k} > \frac{n + nc_k}{1 - kc_k} = v \tag{4.26}$$

If $m < k$, then the above simple arguments do not apply. We have, for an infinite sequence j, $\exists (\mathbf{q}^j, p^j) \in \mathbb{Z}_m^{n+1}$ such that

$$\max\left(e^{t^j} |\langle \mathbf{q}^j, \mathbf{f}(\mathbf{x})\rangle + p^j|, \quad e^{-t_i^j}|q_i^j|\right) \le e^{-d_k t^j}, \quad 1 \le i \le n. \tag{4.27}$$

By assumption $t_i^j \ge d_k t^j$ for at least k values of i. For any such i, we derive from (4.27)

$$|q_i^j| \le e^{t_i^j - d_k t^j}, \quad \text{if } t_i^j \ge d_k t^j. \tag{4.28}$$

From Lemma 2.4 we get that

$$|q_i^j|_+ \le e^{t_i^j - d_k t^j}, \quad \text{if } t_i^j \ge d_k t^j. \tag{4.29}$$

Define for each j the following two index sets:

$$I_1^j = \{i \mid q_i^j \ne 0\}, \quad I_2^j = \{i \mid t_i^j \ge d_k t^j\}. \tag{4.30}$$

By definition

$$\Pi_+(\mathbf{q}^j) = \Pi_{i \in I_1^j} |q_i^j| = \Pi_{i \in I_1^j} |q_i^j|_+ \tag{4.31}$$

Obviously $I_1^j \subset I_2^j$ and this is where the assumption $m < k$ plays a role. Hence

$$\Pi_+(\mathbf{q}^j) = \Pi_{i \in I_1^j} |q_i^j|_+ \le \Pi_{i \in I_2^j} |q_i^j|_+. \tag{4.32}$$

Now we study $\Pi_{i\in I_2^j}|q_i^j|_+$. Denote by b the number of elements in I_2^j. Immediately we get $b \geq k$ from the assumption of the lemma that $t_i^j \geq t^j$ for at least k values of i. Moreover $b \geq k > m$. Hence

$$\Pi_{i\in I_1^j}|q_i^j|_+ \leq e^{t^j - bd_k t^j}. \tag{4.33}$$

Elementary algebra shows that $e^{t^j-bd_kt^j} \leq e^{t^j-kd_kt^j}$. Hence

$$\Pi_{i\in I_1^j}|q_i^j|_+ \leq e^{t^j - kd_k t^j}. \tag{4.34}$$

As a result of the above argument, we have

$$\Pi_+(\mathbf{q}^j) \leq e^{t^j-kd_kt^j}. \tag{4.35}$$

In addition, from (4.27) we have

$$|\langle \mathbf{q}^j, \mathbf{f}(\mathbf{x})\rangle + p^j| \leq e^{-t^j-d_kt^j}. \tag{4.36}$$

From (4.35) and (4.36) we get $\omega^\times(\mathbf{f}(\mathbf{x})) \geq \frac{n+nd_k}{1-kd_k} > v$. Combining the two cases ($m \geq k$ and $m < k$), we see that $\omega^\times(\mathbf{f}(\mathbf{x}) \geq \frac{n+nd_k}{1-kd_k} > v$, $\forall \mathbf{x} \in B \cap$ supp μ, as desired. □

Theorem 4.6. *Let μ be a Federer measure on a Besicovitch metric space X, $\mathcal{L}$ a hyperplane of $\mathbb{R}^n$ and let $\mathbf{f} : X \to \mathcal{L}$ be a continuous map such that $(\mathbf{f}, \mu)$ is good and nonplanar in $\mathcal{L}$. Then the following statements are equivalent for $v \geq n$:*

1. $\{\mathbf{x} \in$ *supp* $\mu \mid \mathbf{f}(\mathbf{x}) \notin W_u^\times\}$ *is nonempty for any $u > v$;*
2. $\omega^\times(\mathbf{f}_*\mu) \leq v$*;*
3. *Condition 4.1 holds for R satisfying* (4.11)*, or equivalently, Condition 4.4 holds for R_0 satisfying* (4.15)*.*

Proof. Suppose the second statement holds. Then the set in the first statement has full measure and is therefore nonempty.

If the third statement holds, then since μ is Federer and $(\mathbf{f}, \mu)$ is good, we can conclude that $\mu - a.e.\ x \in X$ has a neighborhood V such that μ is (C, α)-good and D-Federer on V for some $C, D, \alpha > 0$. Choose a ball $B = B(x, r)$ with positive measure such that the dilated ball $\tilde{B} = B(x, 3^{n+1}r)$ is contained in V. For any $\mathbf{w}$, each of the coordinates of $g_{\mathbf{t}}u_{\mathbf{f}}\mathbf{w}$ is expressed as linear combination of $1, f_1, \ldots, f_n$ from (4.6). By applying an elementary property, see e.g. [8, Lemma 4.1], that whenever $f_1, \ldots, f_N$ are (C, α)-good on a set V with respect to μ, the function $(f_1^2 + \ldots + f_N^2)^{1/2}$ is $(N^{\alpha/2}C, \alpha)$-good on V with respect to μ, we see that the first assumption of Proposition 3.3 is satisfied. The second assumption can be derived from Condition 4.4 by previous discussion concerning the nonplanarity in $\mathcal{L}$. Hence we can apply Proposition 3.3 to establish the second statement.

If the third statement fails to hold, then no ball B intersecting supp μ satisfies Condition 4.4. By Lemma 4.5 $\mathbf{f}(B \cap$ supp $\mu) \subset W_u^\times$ for some $u > v$. This contradicts the first statement. □

From Theorem 4.6 we see that $\omega^\times(\mathcal{L}) \leq \inf\{\omega^\times(\mathbf{y}) \mid \mathbf{y} \in \mathcal{L}\}$ as the first statement of the theorem implies the second one. $\omega^\times(\mathcal{L}) \geq \inf\{\omega^\times(\mathbf{y}) \mid \mathbf{y} \in \mathcal{L}\}$ can be derived from definition. $\omega^\times(\mathcal{L})$ is inherited by its nondegenerate submanifolds as nondegeneracy in $\mathcal{L}$ implies nonplanarity in $\mathcal{L}$ by definition. Therefore

$$\omega^\times(\mathcal{L}) = \omega^\times(\mathcal{M}) = \inf\{\omega^\times(\mathbf{y}) \mid \mathbf{y} \in \mathcal{L}\} = \inf\{\omega^\times(\mathbf{y}) \mid \mathbf{y} \in \mathcal{M}\}$$

and Theorem 1.4 is established.

Besides, Theorem 4.6 establishes that

$$\omega^{\times}(\mathcal{L}) = \sup\{v \mid \text{ Condition 4.4 does not hold}\}. \tag{4.37}$$

For hyperplane $\mathcal{L}$ defined in Theorem 1.5, we embed it into $\mathbb{R}^{n+1}$ as 4.14 by

$$\widetilde{\mathbf{f}}(\mathbf{x}) = (a_1x_1 + \ldots + a_{n-1}x_{n-1} + a_n, x_1, \ldots, x_{n-1}, 1). \tag{4.38}$$

Now we prove Theorem 1.5. Without loss of generality, we suppose from now on that

$$a_1a_2\ldots a_{s-1} \neq 0, \quad \text{ and } a_i = 0 \text{ for } s \leq i \leq n-1. \tag{4.39}$$

Suppose $\mathbf{w} = (p_1, \ldots, p_n, p_0) \in \mathbb{Z}^{n+1}$, then since $j = 1$ the index set J with order $1-1=0$ becomes empty and $\bigwedge^{j-1}(\mathbb{R}^{n+1}) \in \mathbb{Z}$, we have

$$\mathbf{c}(\mathbf{w})_i = \pm\langle \mathbf{e}_i, \mathbf{w}\rangle = \pm p_i \ (1 \leq i \leq n), \quad \mathbf{c}(\mathbf{w})_{n+1} = \pm\langle \mathbf{e}_{n+1}, \mathbf{w}\rangle = \pm p_0. \tag{4.40}$$

We can change the signs of p_i, so we will just use $+$ instead of $\pm$ from now on. Note that

$$\mathbf{c}(\mathbf{w}) = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ p_0 \end{pmatrix} \tag{4.41}$$

and

$$R_0\mathbf{c}(\mathbf{w}) = \begin{pmatrix} a_1 & \\ a_2 & \\ \vdots & I_n \\ a_{n-1} & \\ a_n & \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \\ p_0 \end{pmatrix}. \tag{4.42}$$

Therefore

$$\|R_0\mathbf{c}(\mathbf{w})\| = \left\| \begin{matrix} a_1p_1 + p_2 \\ \vdots \\ a_{s-1}p_1 + p_s \\ a_sp_1 + p_{s+1} \\ \vdots \\ a_{n-1}p_1 + p_n \\ a_np_1 + p_0 \end{matrix} \right\|. \tag{4.43}$$

Unless $p_{s+1} = \ldots = p_n = 0$ and $p_1 \ldots p_s \neq 0$, $\|R_0\mathbf{c}(\mathbf{w})\| \geq \epsilon$ for some positive fixed number ϵ whenever $\mathbf{w}$ is nonzero. In other words the second assumption of Proposition 3.3 is always satisfied except for $\mathbf{w} \in \mathbb{Z}_s^{n+1}$. The above observations coupled with Proposition 3.3 supply a useful tool for establishing upper bounds of multiplicative exponents of hyperplanes described in (4.38). Proof of Theorem 1.5 is based on (4.37):

Proof of Theorem 1.5. We employ the method of proof of Lemma 4.5.

If Condition 4.4 does not hold, $\exists k$ ($1 \le k \le n$, k independent of s) such that for some $d_k > c_k$, $\exists$ an unbounded sequence of t with $t_i \ge d_k t$ for at least k values of i and a sequence of $\mathbf{w} \in \mathbb{Z}_s^{n+1}$, one has

$$\max\left(e^{-t_i}|p_i|, \quad e^t \|R_0\mathbf{c}(\mathbf{w})\|\right) \le e^{-d_k t}, \quad 1 \le i \le n \tag{4.44}$$

$\|R_0\mathbf{c}(\mathbf{w})\|$ is defined in (4.43). After reordering, we may assume that $t_i \ge d_k t$ when $1 \le i \le k$. Consequently we have

$$|p_i|_+ \le e^{t_i - d_k t}, \qquad 1 \le i \le k, \tag{4.45}$$

$$\Pi_+(\mathbf{p}) \le |p_1|_+ \dots |p_k|_+ < e^{t - k d_k t}, \tag{4.46}$$

$$\|R_0\mathbf{c}(\mathbf{w})\| \le e^{t - d_k t}. \tag{4.47}$$

Hence for some $u > v$

$$\|R_0\mathbf{c}(\mathbf{w})\| < \Pi_+(\mathbf{p})^{-u/n}. \tag{4.48}$$

Note that on the other hand by our assumption

$$\Pi_+(\mathbf{p}) = |p_1 p_2 \dots p_s|. \tag{4.49}$$

Hence (4.44) is equivalent to: $\exists$ an infinite sequence of $(p_1, p_2, \dots, p_s, p_0) \in \mathbb{Z}_s^{s+1}$ such that for some $u > v$ we have

$$\left\| \begin{matrix} a_1 p_1 + p_2 \\ a_2 p_1 + p_3 \\ \vdots \\ a_{s-1} p_1 + p_s \\ a_n p_1 + p_0 \end{matrix} \right\| < |p_1 p_2 \dots p_s|^{-u/n}. \tag{4.50}$$

By assuming $\|p_{i+1} + a_i p_1\| \le 1$ for $1 \le i \le s-1$, we deduce that $|p_i| \asymp |p_1|$ for $1 \le i \le s$. Thus (4.50) is equivalent to : $\exists$ a sequence of $(p_1, p_2, \dots, p_s, p_0) \in \mathbb{Z}_s^{s+1}$ with $|p_1|$ unbounded such that for some $u > v$

$$\left\| \begin{matrix} a_1 p_1 + p_2 \\ a_2 p_1 + p_3 \\ \vdots \\ a_{s-1} p_1 + p_s \\ a_n p_1 + p_0 \end{matrix} \right\| \prec |p_1|^{-su/n}, \tag{4.51}$$

where $\prec$ implies some constant dependent on $\mathbf{a}$.

According to (1.12), $\sigma(\mathbf{a})$ is exactly $\frac{s}{n} \sup\{v \mid (4.51) \text{ holds}\}$. Therefore by (4.37) $\omega^\times(\mathcal{L}) = \max\left(n, \dfrac{n}{s}\sigma(\mathbf{a})\right)$. Theorem 1.5 is proved. $\square$

5. A Special Case

We consider a special class of hyperplanes whose multiplicative Diophantine exponents can be obtained in an elementary manner:

Theorem 5.1. *Let $\mathcal{L}$ be a hyperplane in $\mathbb{R}^n$ parameterized by*

$$\mathcal{L} = \left\{(x_1, x_2, \dots, x_{n-1}, a) \,\middle|\, (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\right\} \tag{5.1}$$

Then $\omega^\times(\mathcal{L}) = n\sigma(a)$.

This is a special case of Theorem 1.5 with $s = 1$ and $a_i = 0$ for $1 \le i \le n-1$.

Proof. For an arbitrary $\mathbf{y} = (x_1, x_2, \ldots, x_{n-1}, a) \in \mathcal{L}$, if we approximate it by $\mathbf{q} \in \mathbb{Z}^n$ of the special form $(0, \ldots, 0, q_n)$, we see from (1.7) that

$$\omega^{\times}(\mathbf{y}) \geq n\sigma(a), \quad \forall \mathbf{y} \in \mathcal{L}. \tag{5.2}$$

Hence $\omega^{\times}(\mathcal{L}) \geq n\sigma(a)$. We proceed to prove that $\omega^{\times}(\mathcal{L}) \leq n\sigma(a)$. Apparently,

$$\Pi_{+}(\mathbf{q}) \geq \|\mathbf{q}\|, \quad \forall \mathbf{q} \in \mathbb{Z}^n, \tag{5.3}$$

hence from (1.1) and (1.7) we get

$$\omega^{\times}(\mathbf{y}) \leq n\omega(\mathbf{y}) \quad \forall \mathbf{y} \in \mathbb{R}^n. \tag{5.4}$$

On the other hand it is known from [4] that $\omega(\mathbf{y}) = \sigma(a)$, *a.e.* $\mathbf{y} \in \mathcal{L}$. Hence

$$\omega^{\times}(\mathbf{y}) \leq n\omega(\mathbf{y}) = n\sigma(a), \ a.e. \ \mathbf{y} \in \mathcal{L} \tag{5.5}$$

Combining (5.2) and (5.5) we have $\omega^{\times}(\mathcal{L}) = n\sigma(a)$. □

Remark 5.2. *It is still an open question whether Theorem 1.4 holds for subspaces of codimension bigger than* 1.

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